Curses and Blessings out of the critical slowing down: the evolution of cumulants in QCD critical regime

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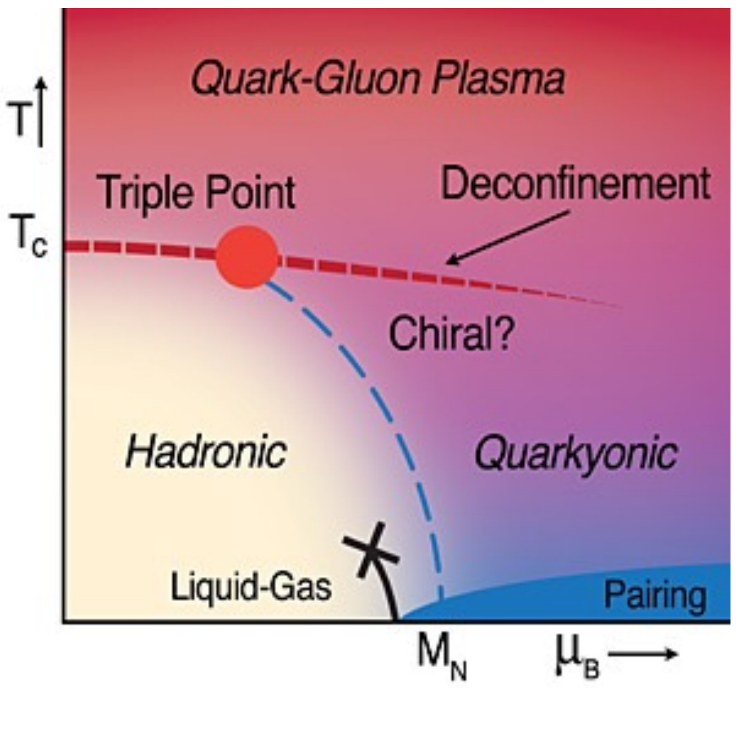
Based on: S. Mukherjee, R. Venugopalan and YY, to appear.

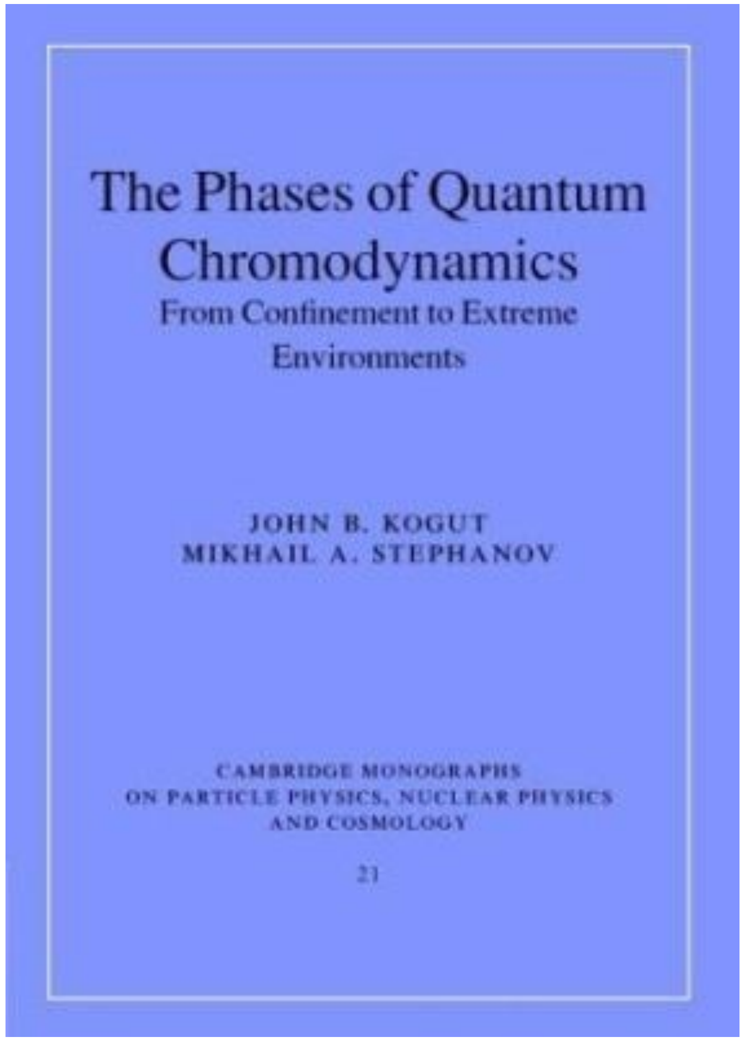
Theory and Modeling for the Beam Energy Scan
RBRC-BNL, Feb. 26-27th

Memory effects from Prof. Stephanov

- Thanks for your supervising, collaboration and conversation.

- Such memory will never be washed out.



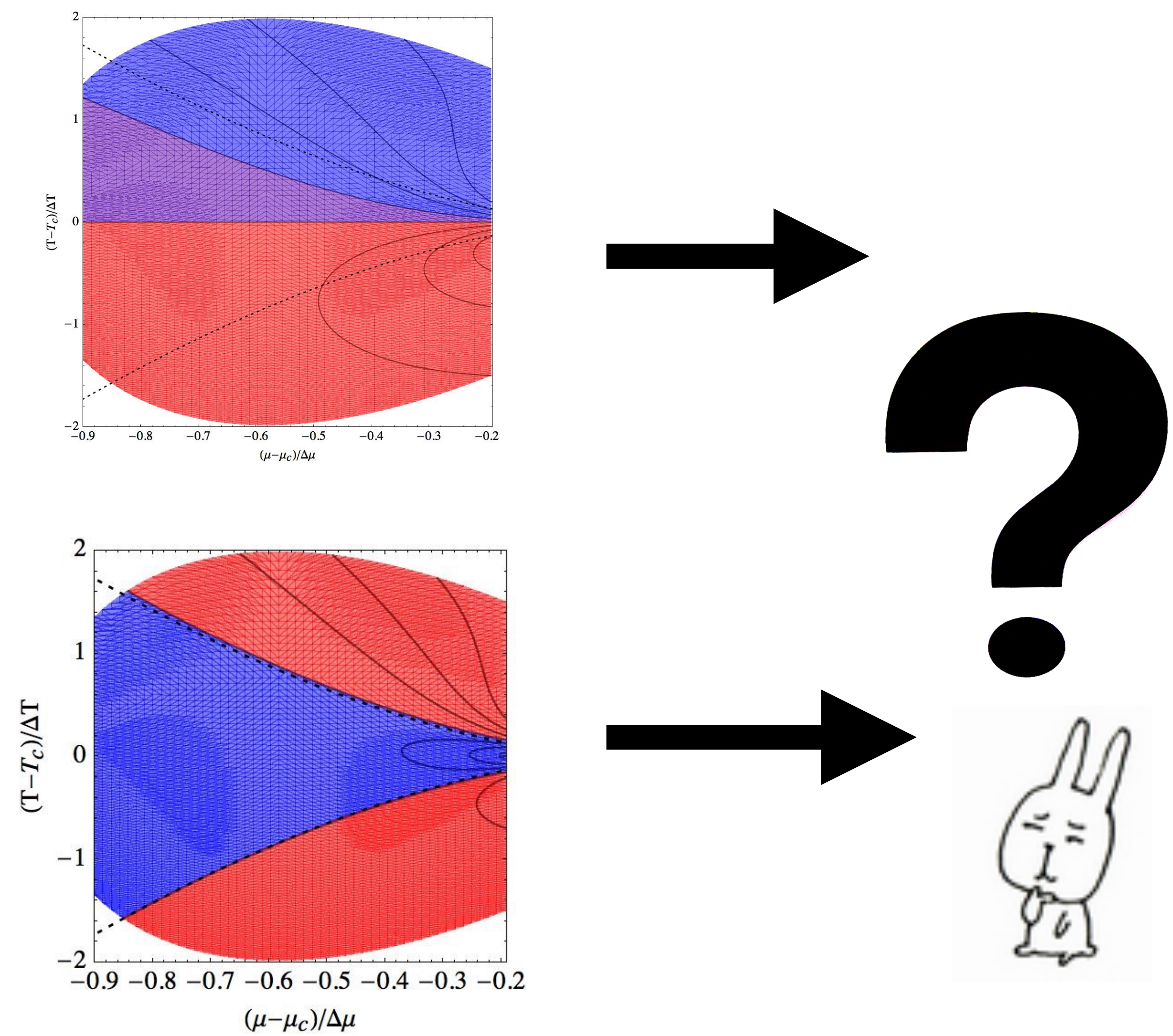


Motivations

- Why cumulants: cumulants, in particular non-Gaussian cumulants are important observables for search for QCD critical point.

- Why real time evolution: fireball only spends a finite time in critical regime, soft-mode in responsible for critical fluctuations is not in equilibrium with the medium.

- Why evolution of the skewness and kurtosis: the sign of them are indefinite. Even a qualitative understanding of their beam energy dependence requires taking memory effects into consideration.



This talk

- Purpose: understand how memory effects would affect the evolution of cumulants, in particular the non-Gaussian ones. Understand the implication of such memory effects for detecting QCD critical point.

- We will focus on the evolution of cumulants(the mean, variance, skewness and kurtosis) of sigma-field in critical regime.

- We will restrict ourselves to the cross-over side of the critical regime but will take universal non equilibrium dynamics into account.

Outline

- Part I: The evolution equations for cumulants.

- Part II: Evolution of cumulants in QCD critical regime.

- Part III: Implications on search for QCD critical point.

Part I: The evolution equations for cumulants.

Moments(cumulants) of σ-field

- We consider zero moment mode of order parameter field σ-field: $\sigma \equiv \frac{1}{V}\int d^3x\, \sigma(\mathbf{x})$.

- Given the probability distribution $P(\sigma;\tau)$, we have (time-dependent) moments

$$\bar{\sigma}(\tau) \equiv \langle \sigma \rangle \,, \qquad \kappa_2(\tau) \equiv \langle (\delta\sigma(\tau))^2 \rangle \,, \qquad \kappa_3(\tau) \equiv \langle (\delta\sigma(\tau))^3 \rangle \,,$$

$$\kappa_4(\tau) \equiv \langle (\delta\sigma(\tau))^4 \rangle - 3\kappa_2^2(\tau) \qquad \delta\sigma \equiv \sigma - \bar{\sigma}(\tau) \,.$$

- We define Skewness and Kurtosis which are independent of the normalization of σ-field(but depends on the volume of the system):

$$S(\text{Skewness}) \equiv \frac{\kappa_3}{\kappa_2^{3/2}} \,, \qquad K(\text{Kurtosis}) \equiv \frac{\kappa_4}{\kappa_2^2} \,.$$

Fluctuations in Equilibrium in 3d Ising Model universality class

- Equilibrium distribution $P_0(\sigma) \sim \exp\left(-V\Omega_0(\sigma)/T\right)$ with the free-energy(density)($m_\sigma^{-1} \equiv \xi_{\rm eq}$)

$$\Omega_0(\sigma) = \frac{1}{2} m_\sigma^2 (\sigma - \sigma_0)^2 + \frac{\lambda_3}{3} (\sigma - \sigma_0)^3 + \frac{\lambda_4}{4} (\sigma - \sigma_0)^4 \;,$$

- Universality scaling($V_4 \equiv V/T$):

$$\sigma_0 \sim \tilde{\sigma}_0 T (T\xi)^{-1/2} \,, \qquad \lambda_3 \sim \tilde{\lambda}_3 (T\xi)^{-3/2} \,, \qquad \lambda_4 \sim \tilde{\lambda}_4 (T\xi)^{-1} \,.$$

- The equilibrium moments are given by:

$$\kappa_2^{\rm eq} = \frac{\xi_{\rm eq}^2}{V_4}\left[1 + \mathcal{O}(\epsilon^2)\right] \;, \qquad \kappa_3^{\rm eq} = -\frac{2\xi_{\rm eq}^6}{V_4^2}\lambda_3 \,, \qquad \kappa_4^{\rm eq} = \frac{6\xi_{\rm eq}^8}{V_4^3}\left[2(\lambda_3\xi)^2 - \lambda_4\right] \;,$$

- It is convenient to rescale the quantity by the width of the equilibrium distribution, we observe hierarchy ϵ for different cumulants.

$$b \equiv \sqrt{\frac{1}{V_4 m_\sigma^2}},$$

- For rescaled moments, $\tilde{\kappa}_n \equiv \kappa_n / b^n \,, \qquad n = 2, 3, 4, \ldots \,,$

$$\tilde{\kappa}_2^{\rm eq} = 1 + \mathcal{O}(\epsilon^2) \,, \qquad \tilde{\kappa}_3^{\rm eq} = -2\tilde{\lambda}_3 \epsilon \,, \qquad \tilde{\kappa}_4^{\rm eq} = 6\left[2(\tilde{\lambda}_3)^2 - \tilde{\lambda}_4\right]\epsilon^2 \,, \qquad \epsilon \equiv \sqrt{\frac{\xi_{\rm eq}^3}{V}} \,.$$

The evolution of non-equilibrium $P(\sigma;\tau)$

- Fokker-Planck equation descrbies the relaxation of non-equilibrium distribution $P(\sigma,\tau)$ towards the equilibrium distribution(Hohenberg-Halperin, 1977),

$$\partial_\tau P(\sigma;\tau) = \frac{1}{m_\sigma^2 \tau_{\text{eff}}} \left\{ \partial_\sigma \left[\partial_\sigma \Omega_0(\sigma) + V_4^{-1} \partial_\sigma \right] P(\tilde{\sigma};\tau) \right\}, \qquad \tau_{\text{eff}} \sim \xi^z$$

- The information on the evolution of all cumulants are encoded in Fokker-Planck equation. However, it not easy to gain intuition on how non-Gaussian cumulants evolves by solving it numerically.

- Can one find a a set of equation which directly describe the evolution of cumulants we are interested in$(\bar{\sigma}, \kappa_2, \kappa_3, \kappa_4)$?

A set of equation of cumulants evolution

- We derive, to leading order in $\epsilon = \sqrt{\xi^3/V}$($\xi$ is larger than microscopic scale but smaller than the size of the system), a set of equation from Fokker-Planck equation for $\bar{\sigma}, \kappa_2, \kappa_3, \kappa_4$(S. Mukherjee, R. Venugopalan and YY, to appear.):

$$b^{-1}\partial_\tau \bar{\sigma}(\tau) = -\tau_{\text{eff}}^{-1}\left[(\frac{\bar{\sigma}-\sigma_0}{b})F_1(\bar{\sigma})\right][1+\mathcal{O}(\epsilon)] \ ,$$

$$b^{-2}\partial_\tau \kappa_2(\tau) = -2\tau_{\text{eff}}^{-1}\left[F_2(\bar{\sigma})\tilde{\kappa}_2 - 1\right][1+\mathcal{O}(\epsilon)] \ ,$$

$$b^{-3}\partial_\tau \kappa_3(\tau) = -3\tau_{\text{eff}}^{-1}\left[F_2(\bar{\sigma})\tilde{\kappa}_3(\tau) + \epsilon F_3(\bar{\sigma})\left(\tilde{\kappa}_2(\tau)\right)^2\right][1+\mathcal{O}(\epsilon)] \ ,$$

$$b^{-4}\partial_\tau \kappa_4(\tau) = -4\tau_{\text{eff}}^{-1}\left[F_2(\bar{\sigma})\tilde{\kappa}_4(\tau) + 3\epsilon F_3(\bar{\sigma})\left(\tilde{\kappa}_2(\tau)\tilde{\kappa}_3(\tau)\right) + \epsilon^2 F_4\left(\tilde{\kappa}_2\right)^2\right][1+\mathcal{O}(\epsilon)] \ .$$

  $F_n(\bar{\sigma}), n = 1,2,3,4$ are polynomials of $\bar{\sigma}$ and only depends on the equilibrium properties of the system.

- Derivation is straightforward by substituting σ^n into Fokker-Planck equation and integrate over σ.

The Gaussian limit

- If the equilibrium distribtuion is Gaussian: $\tilde{\Omega}_0(\sigma) = \frac{1}{2}\left(\tilde{\sigma} - \tilde{\sigma}_0\right)^2, (\kappa_2^{\mathrm{eq}}(\tau) = b^2(\tau)\,, \kappa_3^{\mathrm{eq}} = \kappa_4^{\mathrm{eq}} = 0)$, the evolution among cumulants decouple:

$$\partial_\tau \bar{\sigma} = -\tau_{\mathrm{eff}}^{-1}\left[\bar{\sigma}(\tau) - \sigma_0(\tau)\right], \qquad \partial_\tau \kappa_2(\tau) = -2\tau_{\mathrm{eff}}^{-1}\left[\kappa_2(\tau) - k_2^0(\tau)\right],$$

$$\partial_\tau \kappa_3(\tau) = -3\tau_{\mathrm{eff}}^{-1}\kappa_3(\tau)\,, \qquad \partial_\tau \kappa_4(\tau) = -4\tau_{\mathrm{eff}}^{-1}\kappa_4(\tau)\,.$$

- Simple relaxation equaiton, any non-Gaussian cumulants will be damped.

- If one defines non-equilibrium correlation length $\xi(\tau) \equiv \sqrt{V_4 \kappa_2(\tau)}$. In the near equilibrium limit, it can be matched to equation used by Berdnikov-Rajagopal:

$$\partial_\tau\left[\xi^{-1}(\tau)\right] = -\tau_{\mathrm{eff}}^{-1}\left[\xi^{-1}(\tau) - \xi_{\mathrm{eq}}^{-1}(\tau)\right].$$

Near equilibrium limit

- If $\sigma \to \sigma_0$ and the deviation from equilibrium of cumulants is small $\delta\tilde{\kappa}_n \equiv \tilde{\kappa}_n - \tilde{\kappa}_n^{\text{eq}}$

$$\partial_\tau \bar{\sigma}(\tau) = -\tau_{\text{eff}}^{-1} \left(\bar{\sigma} - \sigma_0\right) , \qquad b^{-1}\partial_\tau \kappa_2(\tau) = -2\tau_{\text{eff}}^{-1}\delta\tilde{\kappa}_2(\tau) ,$$

$$b^{-3}\partial_\tau \kappa_3(\tau) = -3\tau_{\text{eff}}^{-1} \left[\delta\tilde{\kappa}_3(\tau) + 4\epsilon\tilde{\lambda}_3\delta\tilde{\kappa}_2(\tau)\right] ,$$

$$b^{-4}\partial_\tau \kappa_4(\tau) = -4\tau_{\text{eff}}^{-1}\{ \delta\tilde{\kappa}_4(\tau) + 6\epsilon\tilde{\lambda}_3\delta\tilde{\kappa}_3 - 12\epsilon^2 \left[(\tilde{\lambda}_3)^2 - \tilde{\lambda}_4\right] \delta\tilde{\kappa}_2\} .$$

- Coupled evolution. Lower moments will be relaxed back to the equilibrium first.

Summary of Part I:

- We have derived a set of equations for the evolution of cumulants.
- The evolution of non-Gaussian cumulants are coupled to the Gaussian cumulant and the mean.
- We now apply it to the QCD critical regime.

Phenomenological inputs

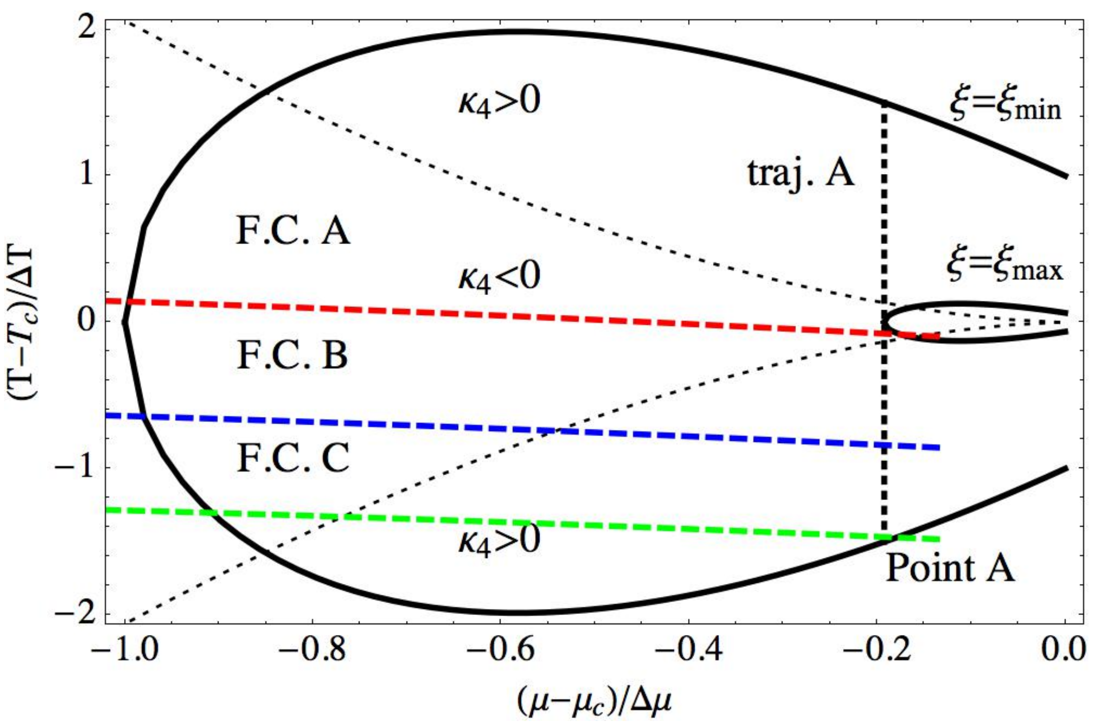
- We will apply our equations to study the evolution of cumulants in QCD critical regime. We therefore need phenomenological inputs.
- We define the scaling regime with the criterion: $\xi_{min} < \xi_{\rm eq} < \xi_{\rm max}$ and to be specific, we will take $\xi_{\rm max}/\xi_{\rm min} = 3$ below.
- The equilibrium distribution is known in Ising variables r, h. We need to map them to QCD variables T, μ_B.(Non-universal, major uncertainty). We use linear mapping with $\Delta T, \Delta\mu$ the width of critical regime in QCD phase diagram.

$$\frac{T - T_c}{\Delta T} = -\frac{h}{\Delta h}, \qquad \frac{\mu - \mu_c}{\Delta \mu} = -\frac{r}{\Delta r}.$$

- Parametrization of $\tau_{\rm eff}$ on $\xi_{\rm eq}$ is universal:$\tau_{\rm rel}$ the relaxation time at $\xi = \xi_{\rm min}$.

$$\tau_{\rm eff} = \tau_{\rm rel}\left(\frac{\xi}{\xi_{\rm min}}\right)^z.$$

and we use Model H, $z = 3$.



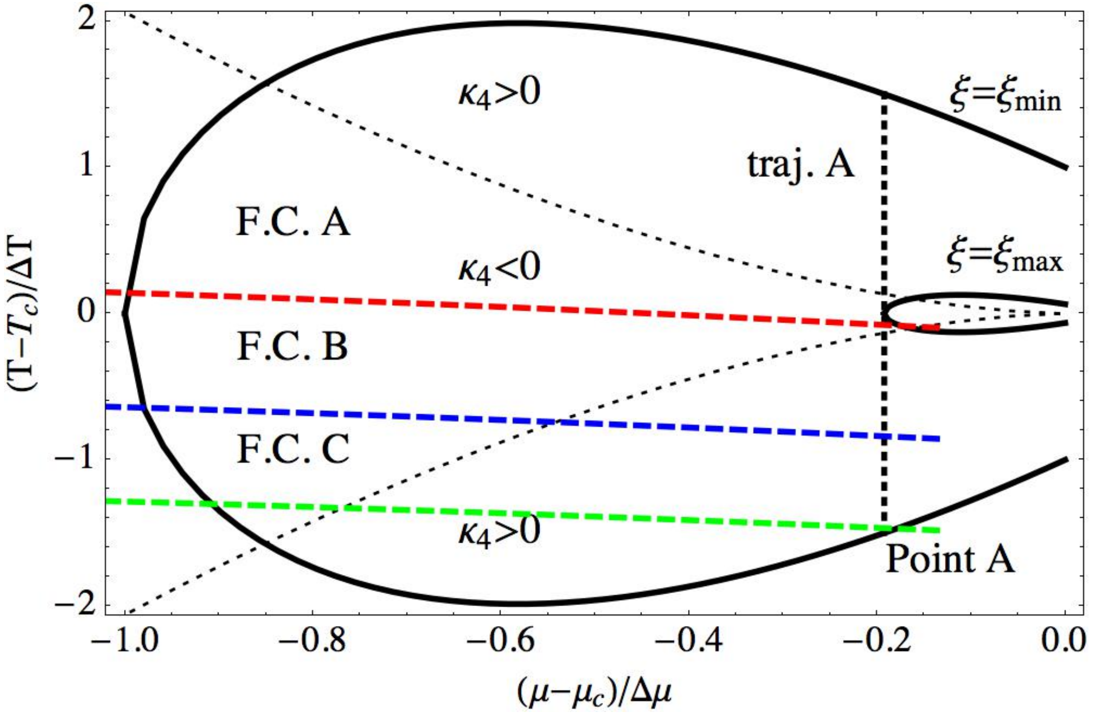
Trajectory

- We will assume for each trajectory, the μ_B of the fireball is constant. It would then be corresponding to a vertical line in the critical regime due to our mapping relation.
- Along each trajectory, we parametrize the evolution of volume and temperature by expansion rate $n_V = 3$ and speed of sound c_s^2:

$$\frac{V(\tau)}{V_I} = \left(\frac{\tau}{\tau_I}\right)^{n_V}, \qquad \frac{T(\tau)}{T_I} = (\frac{\tau}{\tau_I})^{-n_V c_s^2},$$

where V_I, T_I are volume and temperature of the system at τ_I, the time when the trajectory hits the boundary of critical regime.

- Initial condition: we will assume $\bar{\sigma}, \kappa_2, \kappa_3, \kappa_4$ equal to their equilibrium value at $\tau = \tau_I$(τ_{eff} is small).



Part II: Evolution of cumulants in QCD critical regime.

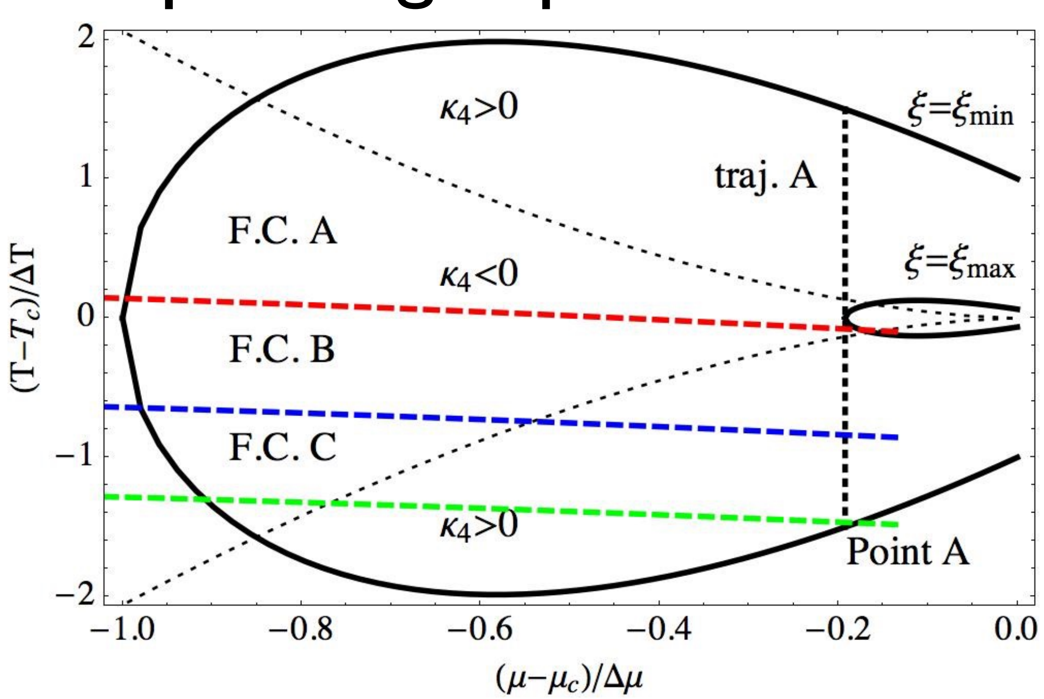
The evolutions

- Only one free parameter: $\tau_{\rm rel}/\tau_I$

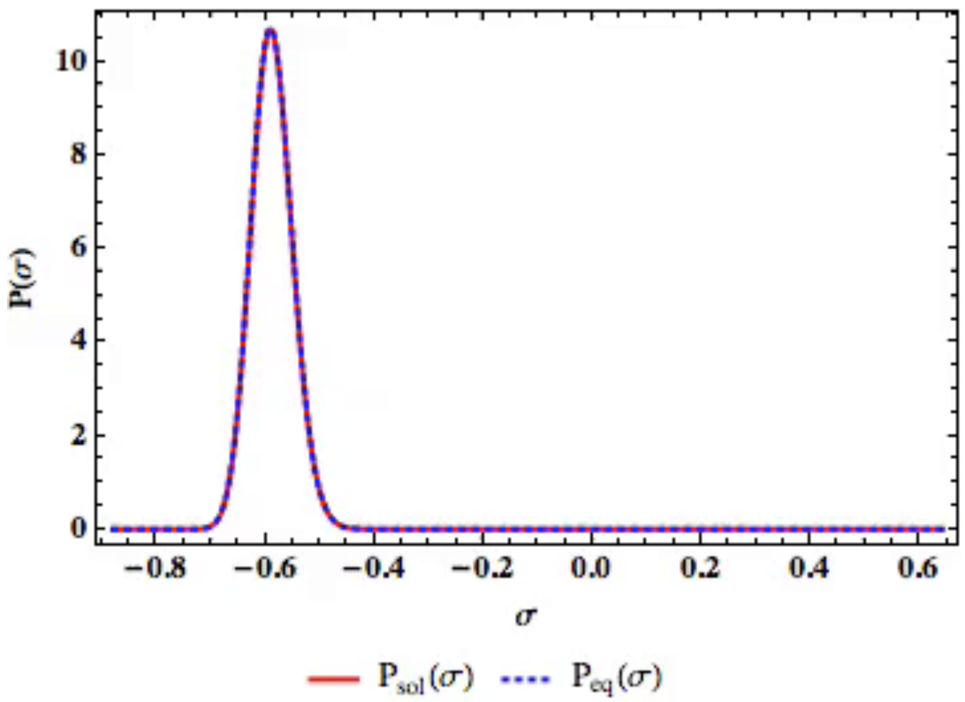
- We have solved evolution equations along trajectories passing through the critical regime.

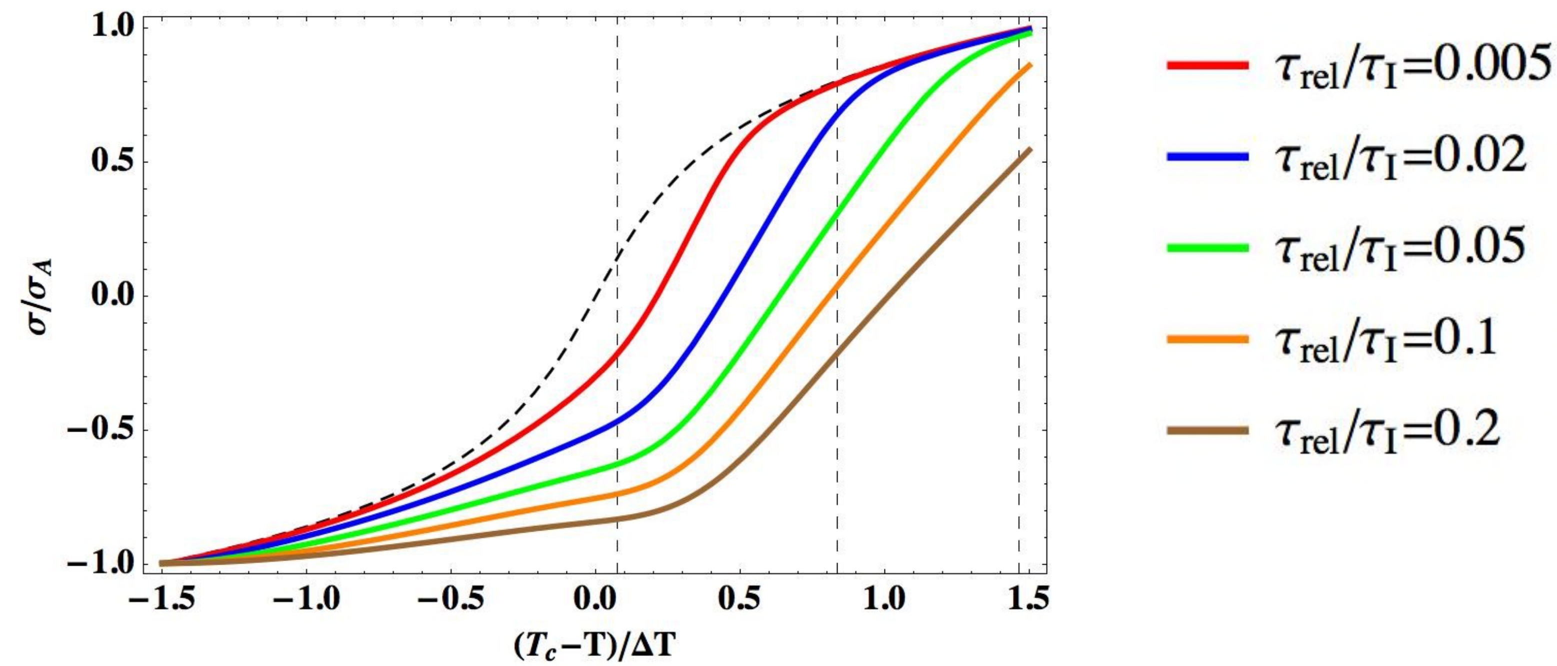
- We label the trajectory crossing the critical regime by the corresponding temperature and will present the non-equilibrium value with different choices of relaxation time.

- We rescale our results by the corresponding equilibrium value at point A.



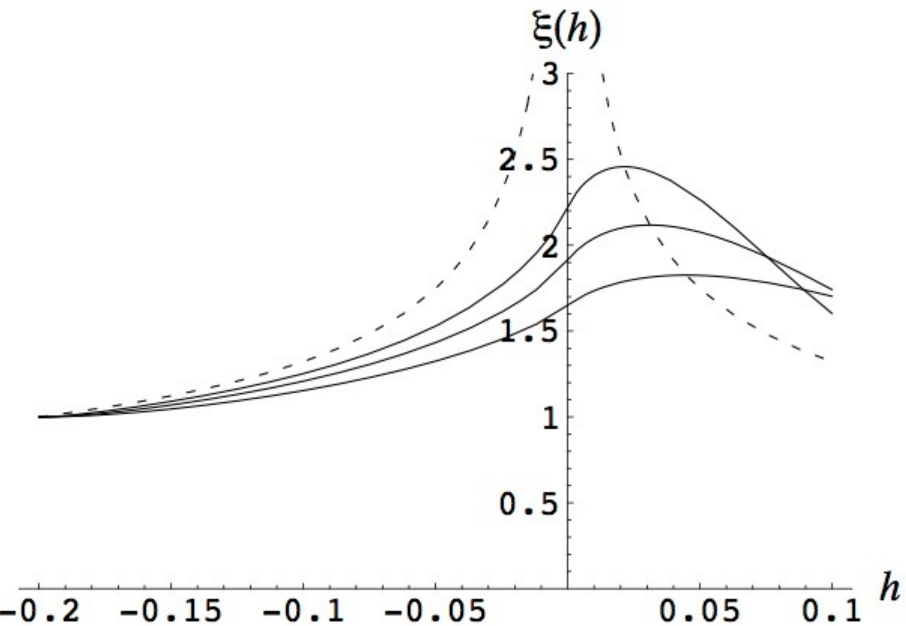
Evolution of magnetization $\bar{\sigma}$



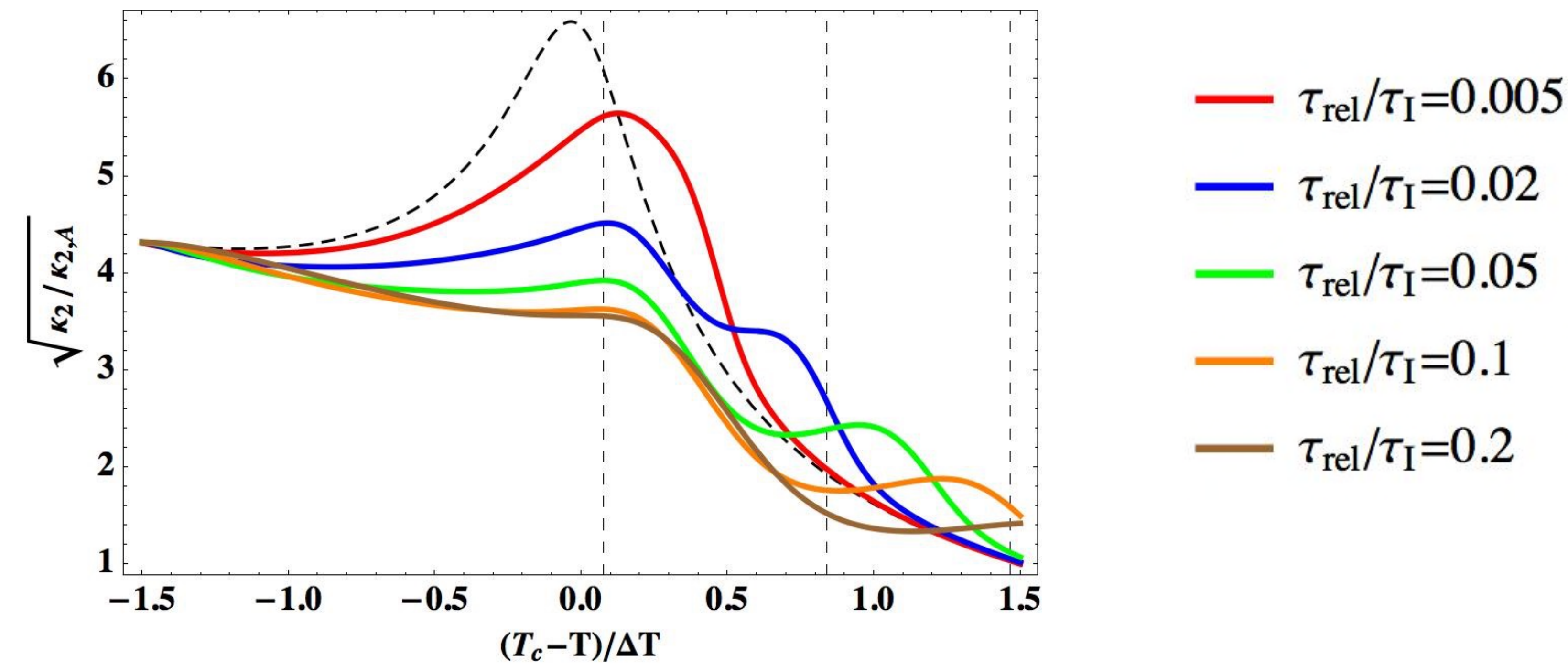


- $\bar{\sigma}$ tend to approach its equilibrium value but still fall behind
- As expected, the slowing down is most visible around Tc where the equilibrium correlation becomes large.

Evolution of Gaussian moment



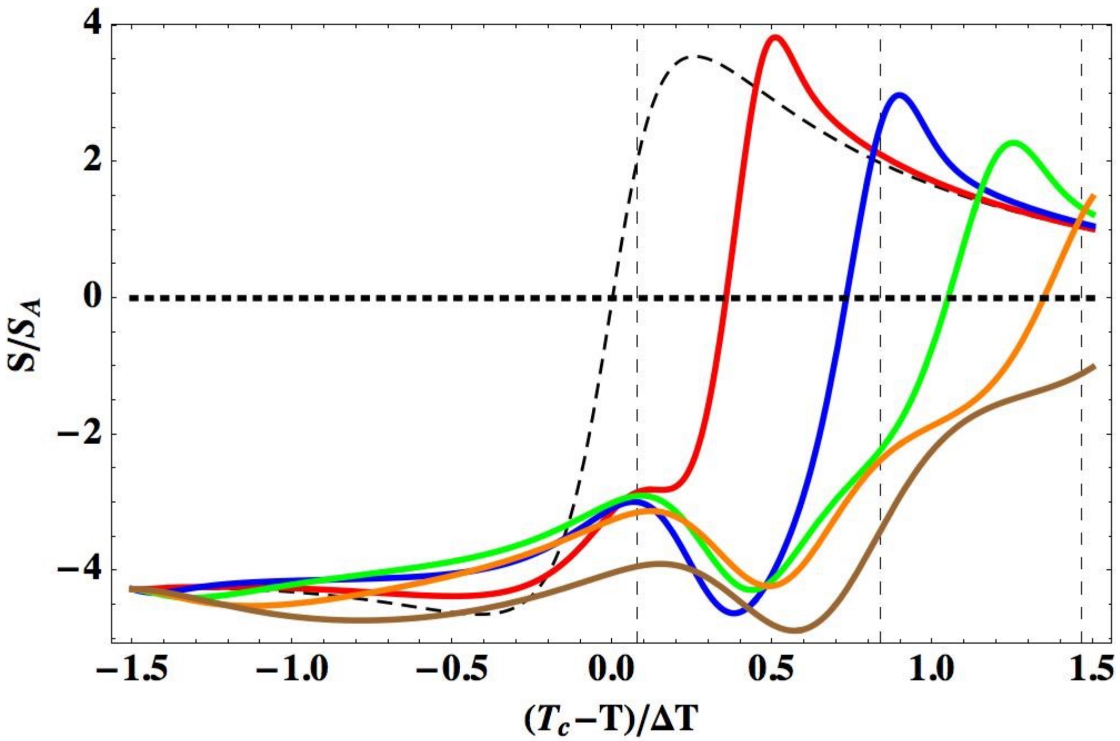
Berdnikov-Rajagopal, 2000

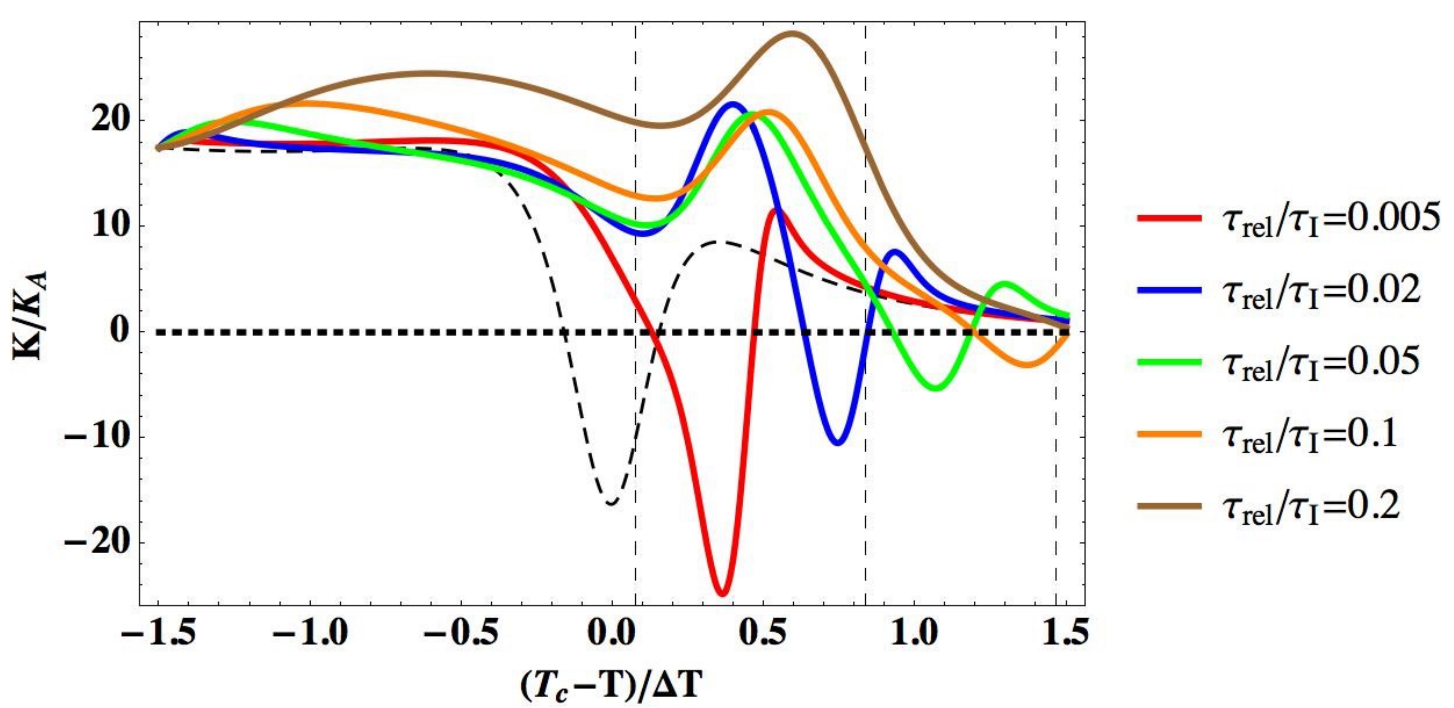


Evolution of variance along a representative trajectory

- The effects of critical slowing down would delay the growth of non-equilibrium length.
- On the other hand, memory effect also protects the memory of the system in critical regime from being completely washed out.
  - Similar to previous results.

Skewness and Kurtosis



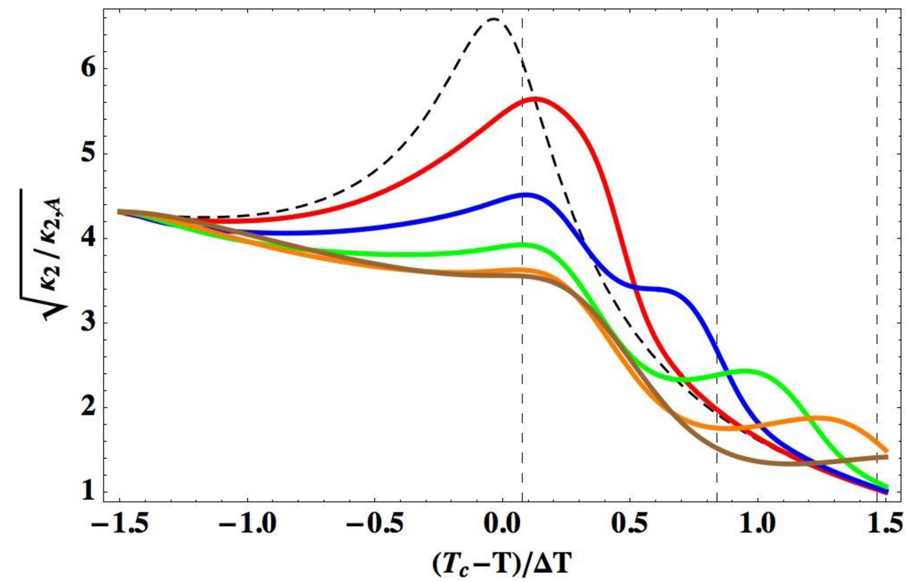


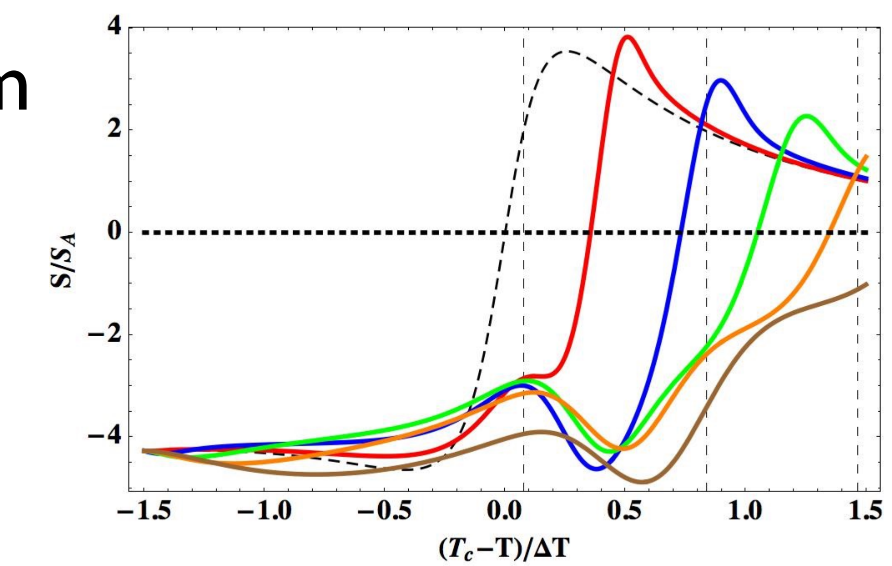
Evolution of skewness and kurtosis along a representative trajectory

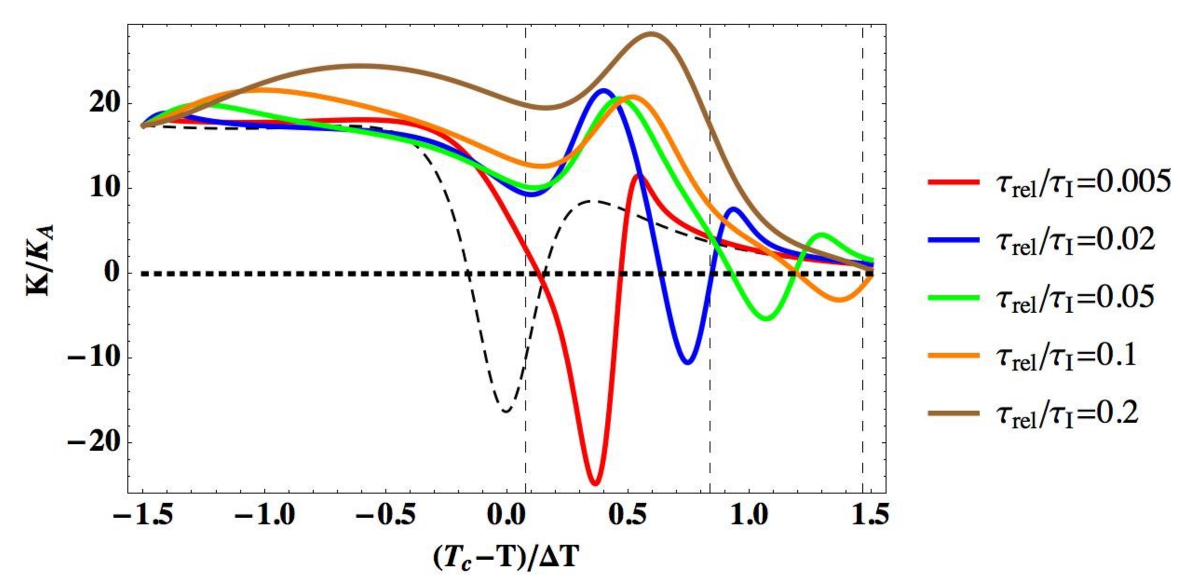
- The evolution of higher cumulants might not follow the equilibrium moments (low moments will affect the evolution of the higher one).
- Depending on the temperature at which you take the snapshot, the non-equilibrium value can be substantially different(including sign) from the equilibrium one.
- Evolution of higher cumulants has a richer pattern(the evolution equations are coupled.)

Quick Synopsis

- Memory effects are important.
- Gaussian cumulants approach equilibrium first, then higher cumulants.
- The tails of evolutions for different relaxation times exhibit possible self-similarity behavior(finite time scaling?).



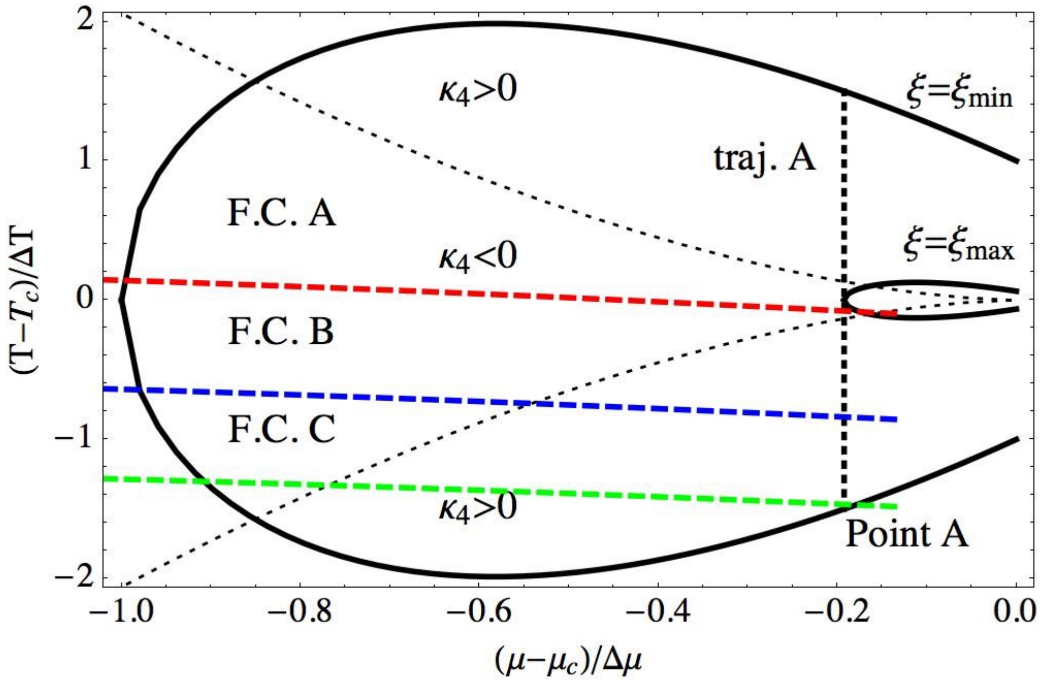




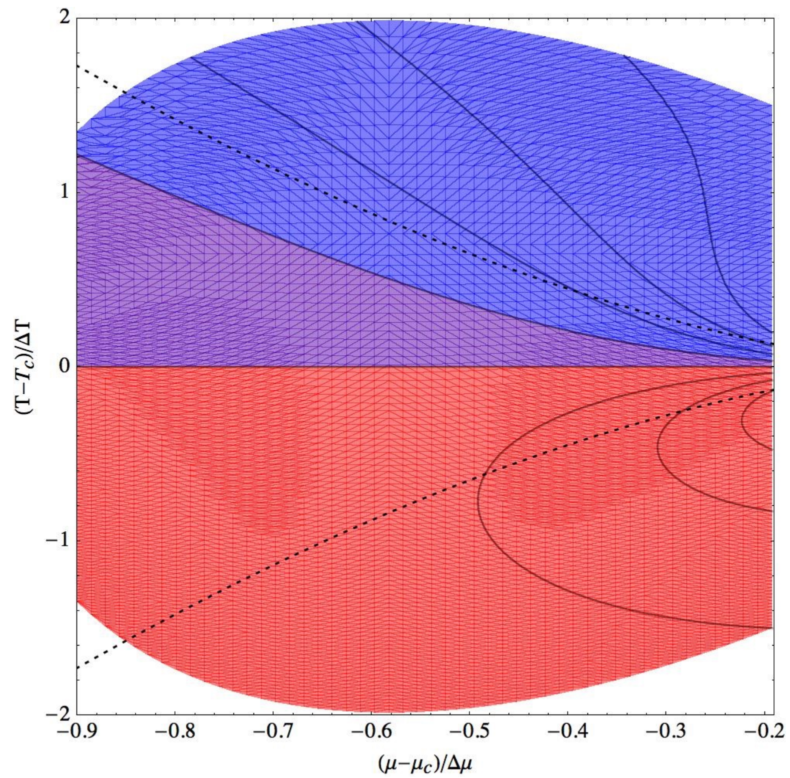
Part III: Implications of results for the search for QCD critical point

Mimicking Beam Energy Scan

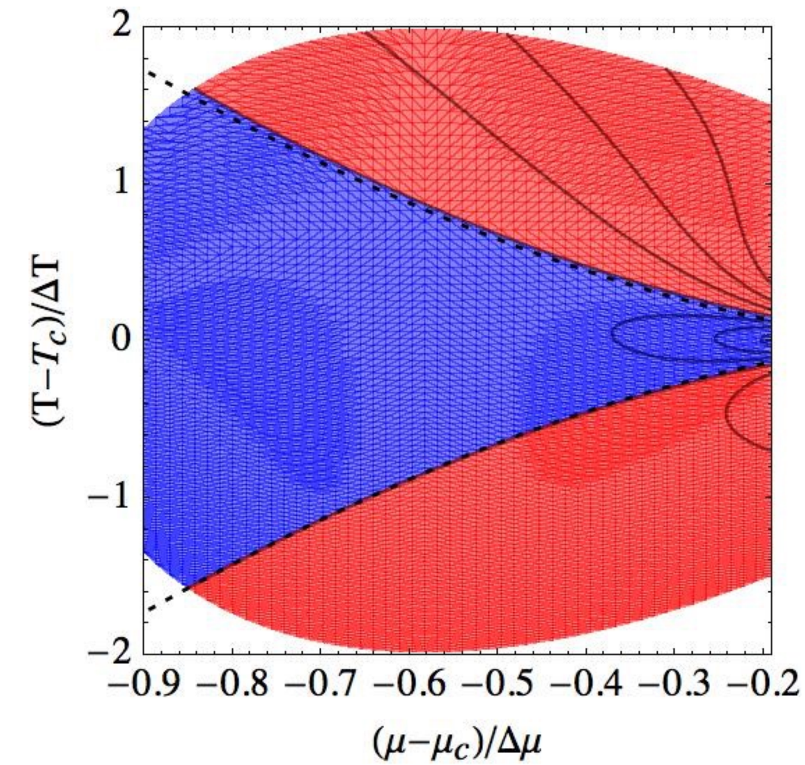
- To mimic the beam energy scan, we also solved the evolution equations for all constant μ trajectories. We therefore obtain non-equilibrium at each point in the critical regime.
- We now examine the memory effects on BES scan.
- We will concentrate on the Skewness and Kurtosis and will start with their most prominent feature: sign.



(Sign of)Equilibrium Skewness and Kurtosis



Skewness



Red >0

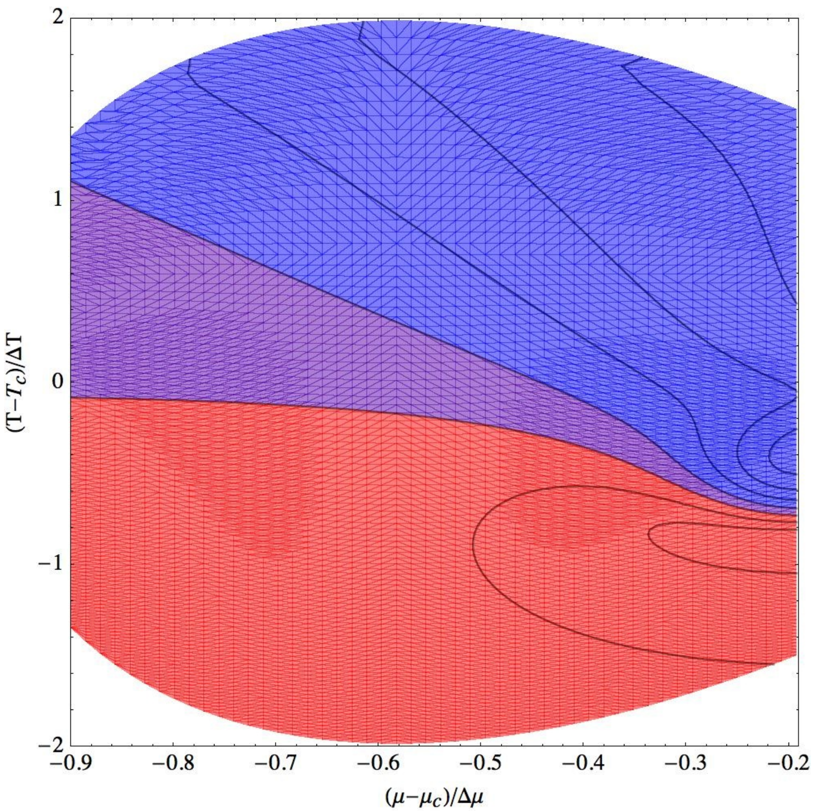
Blue<0

Kurtosis

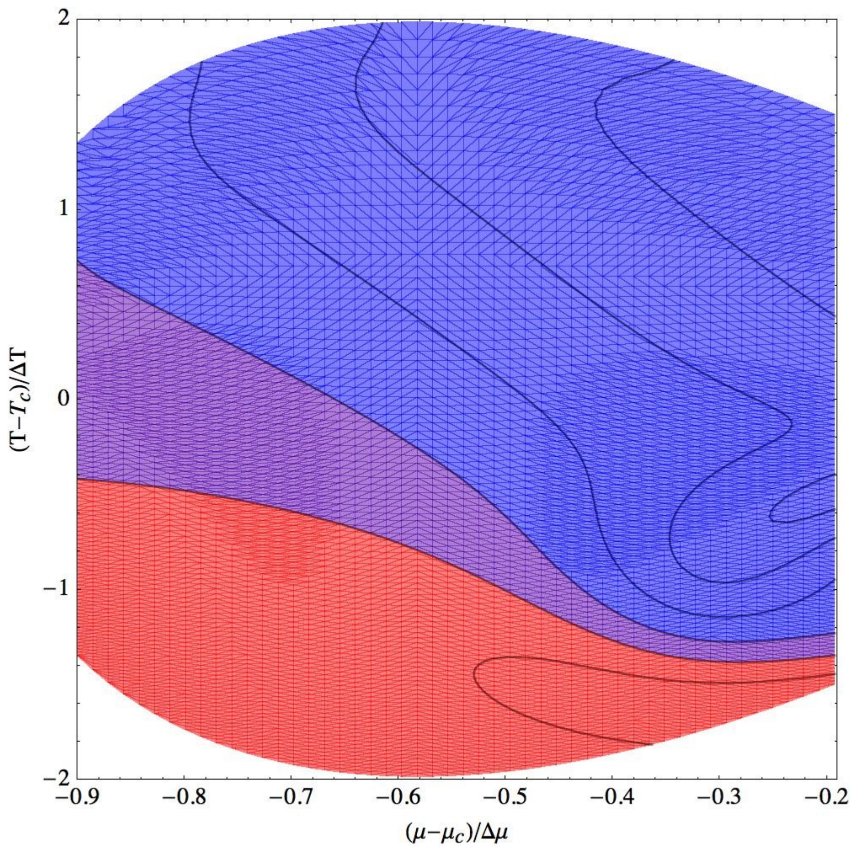
- Following the argument by Stephanov(Phys.Rev.Lett. 102 (2009) 032301), we assume the sign of skewness is positive below cross-over line.

- How would non-equilibrium effects change the above picture?

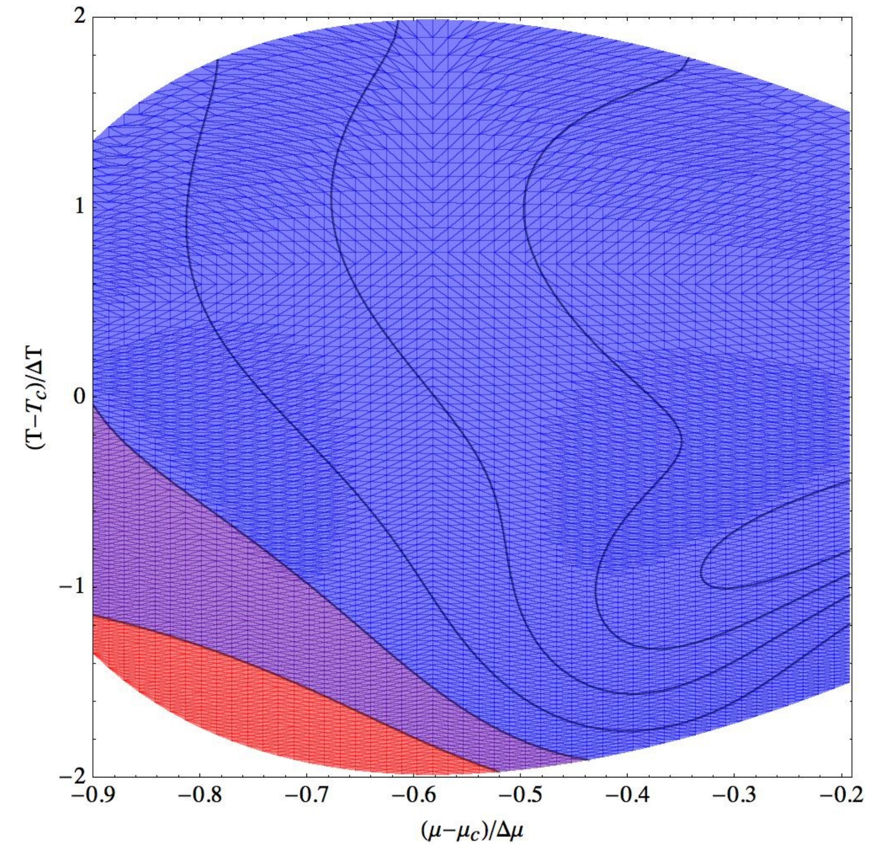
Deformation effects: Skewness



$\tau_{\mathrm{rel}}/\tau_I = 0.02$



$\tau_{\mathrm{rel}}/\tau_I = 0.2$

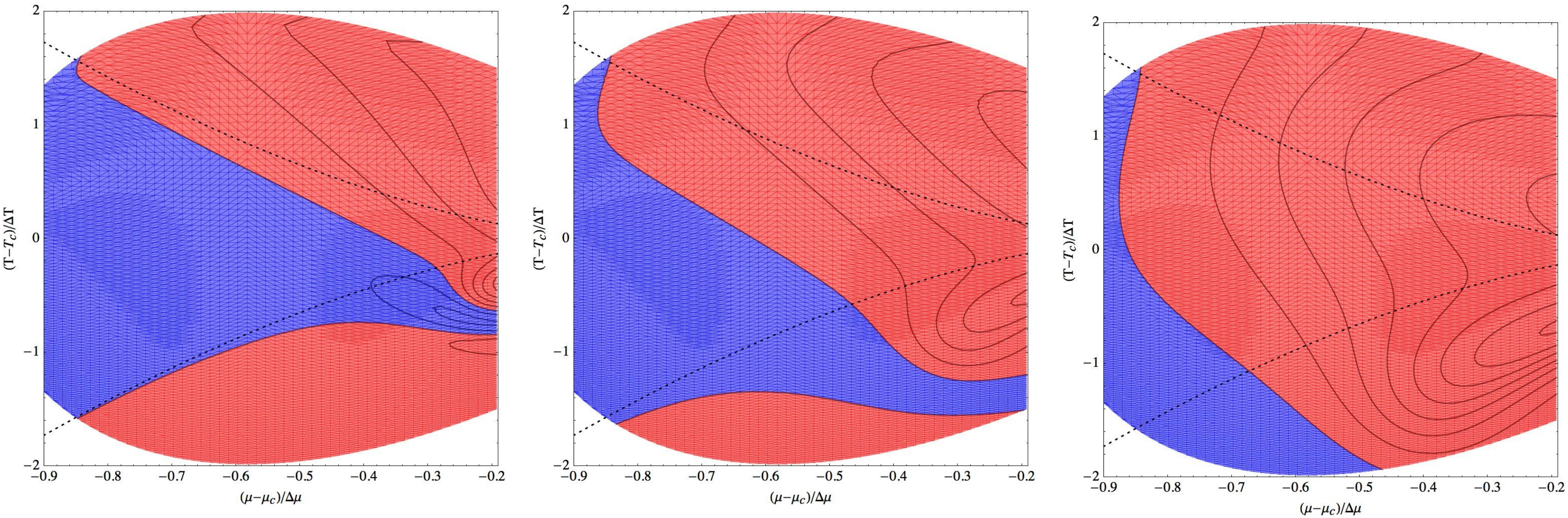


$\tau_{\mathrm{rel}}/\tau_I = 0.5$

Non-equilibrium skewness in critical regime

- Non-equilibrium effects deforms the regime that skewness is positive(negative).
- Non-equilibrium skewness carries the memory from deconfined phase(negative sign).

Deformation effects: Kurtosis



$\tau_{\rm rel}/\tau_I = 0.02$ $\tau_{\rm rel}/\tau_I = 0.2$ $\tau_{\rm rel}/\tau_I = 0.5$

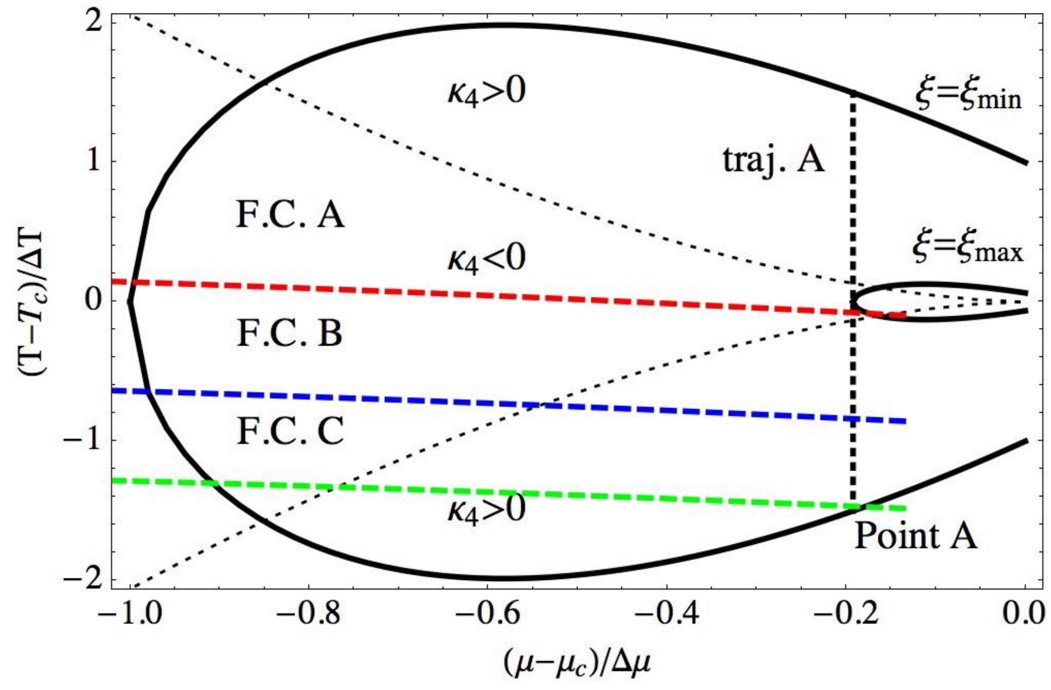
Non-equilibrium kurtosis in critical regime

- Similar for kurtosis. The boundary that kurtosis will change sign also deform.

Skewness and Kurtosis on freeze-out curves

- We now present non-equilibrium results on the freeze-out curves.

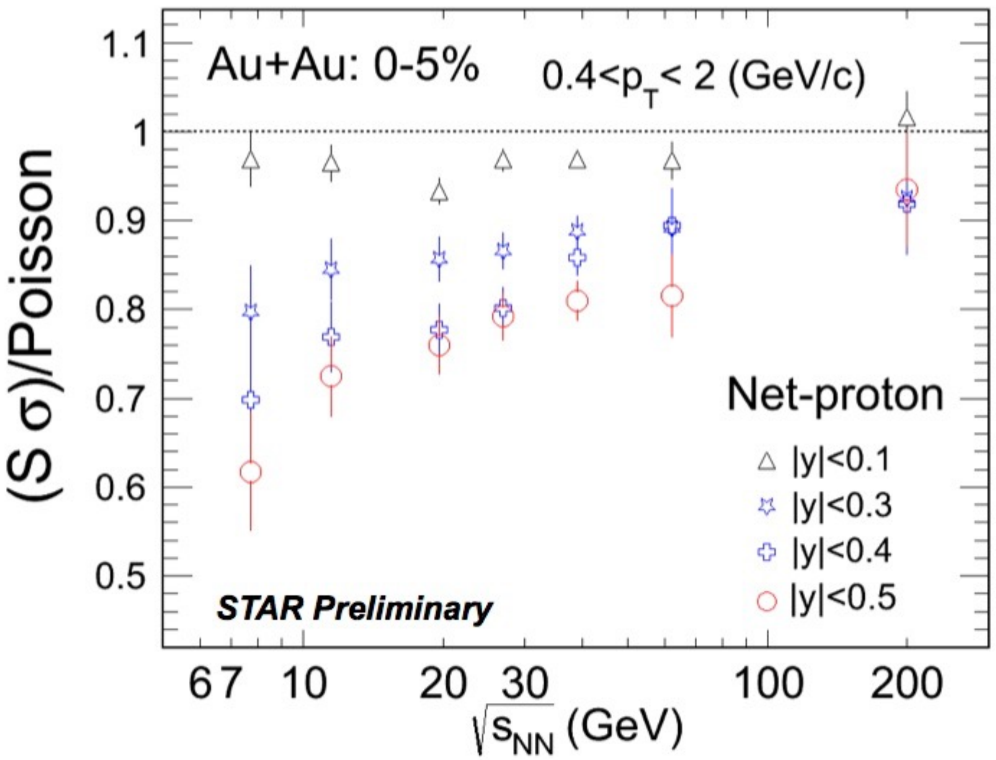
- The relative position between the freeze-out curves and critical regime depends on the location of critical as well as the width of the critical regime.

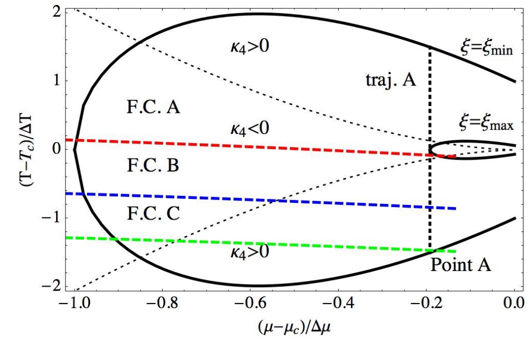


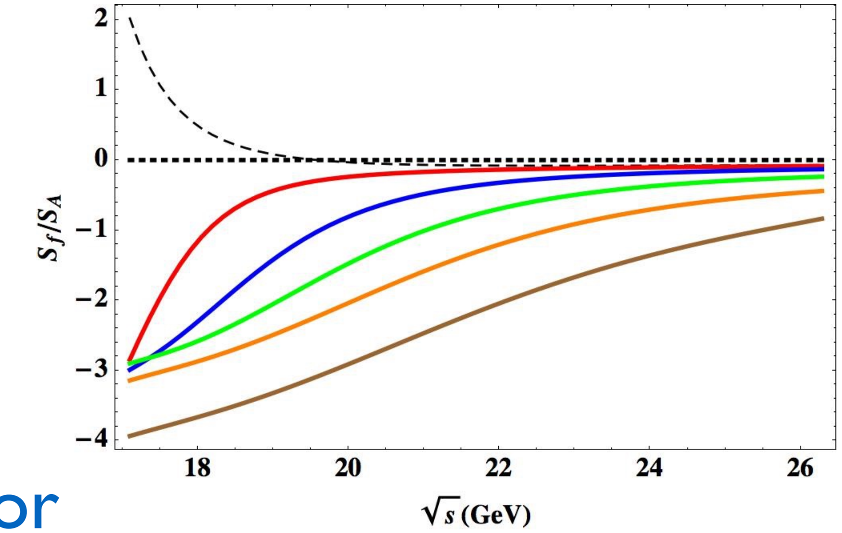
We fix $\mu_c = 300$ MeV, $\Delta\mu = 100$ MeV, $\Delta T/T_c = 1/8$ but take $T_c = 160, 175, 190$ MeV to consider three different relative positions of freeze-out curves. We will convert μ into $\sqrt{s}$ dependence as well.

- Disclaimer: This is neither a prediction nor a fitting. The purpose is to illustrate memory effects.

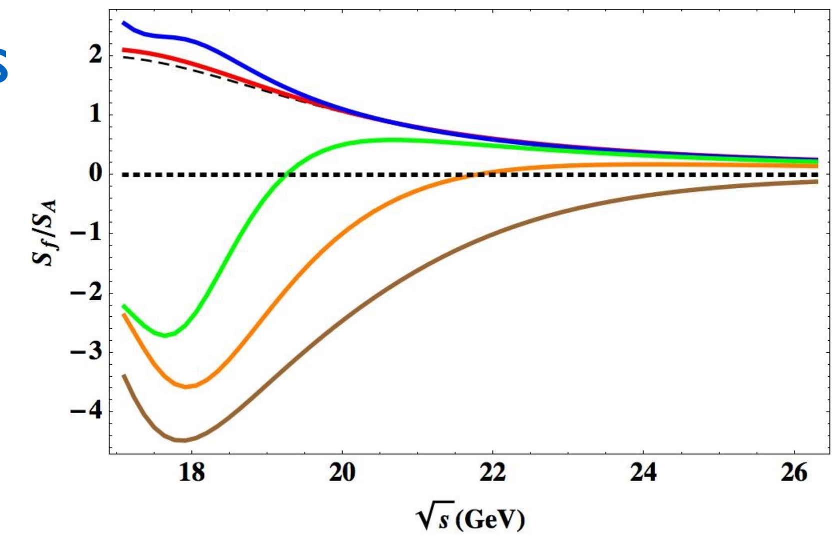
Skewness on freeze-out curves



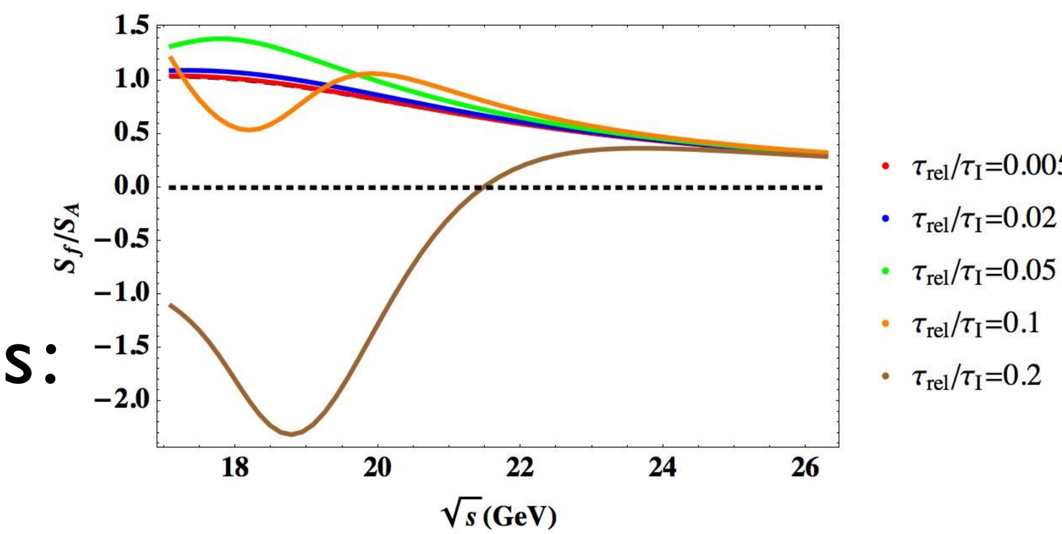




Skewness on f. curves for three different positions of f.curves.

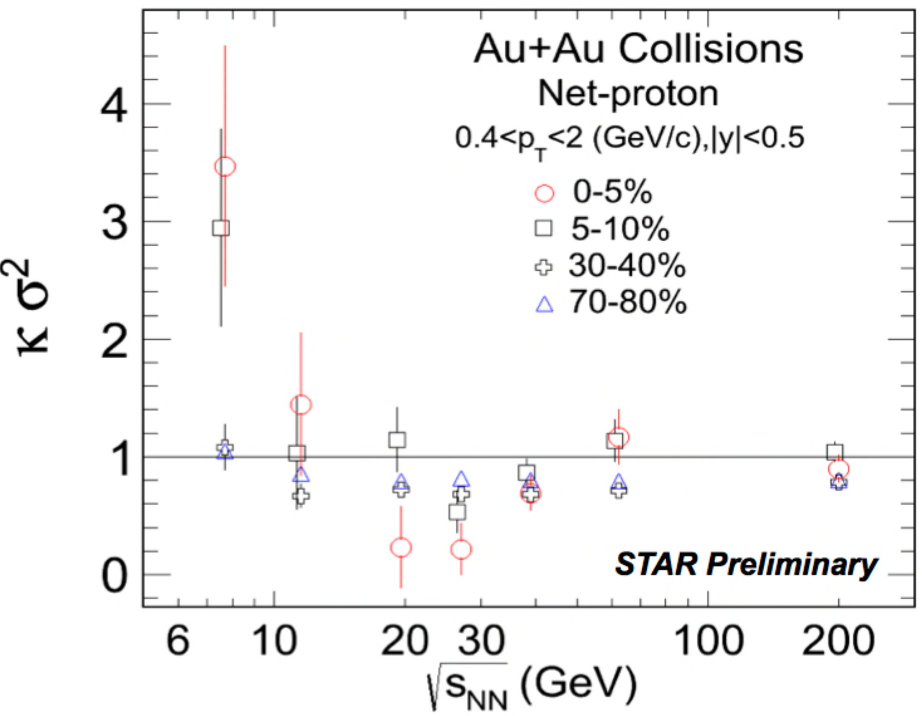


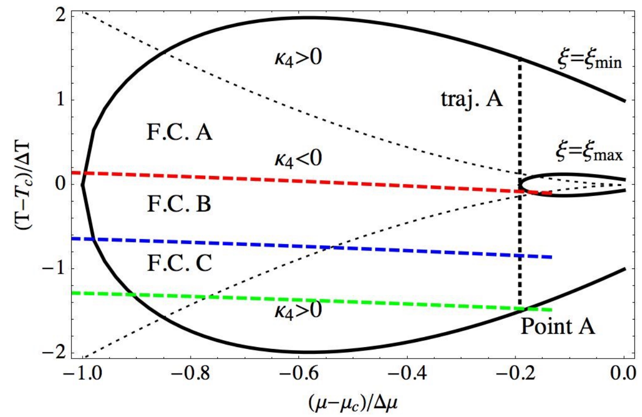
- The behavior of non-equilibrium skewness can be non-monotonous even if the equilibrium skewness is monotonous.



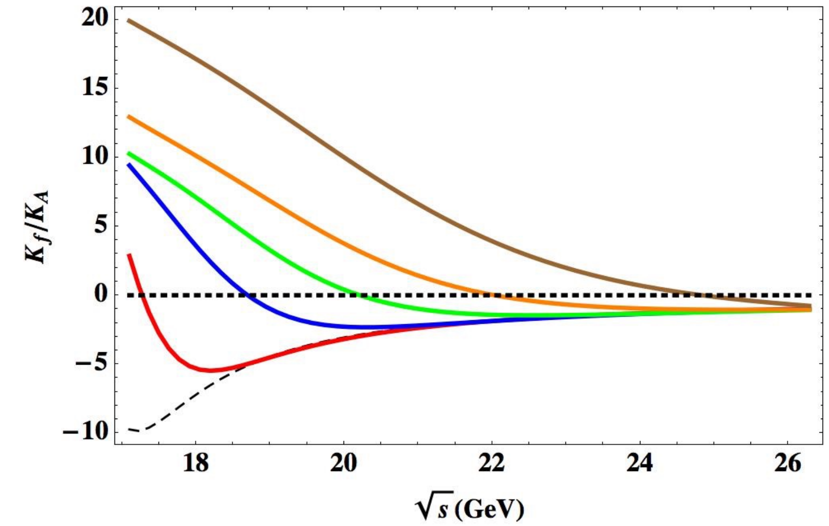
- The sign of non-equilibrium skewness can be opposite to the equilibrium skewness.
- Negative contribution to skewness: memory effects?

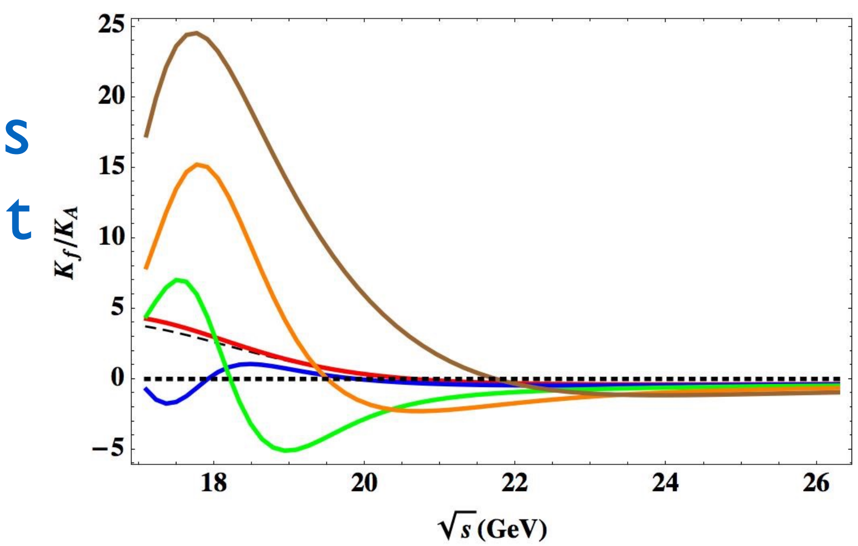
Non-equilibrium Kurtosis(of sigma field) on freeze-out curves

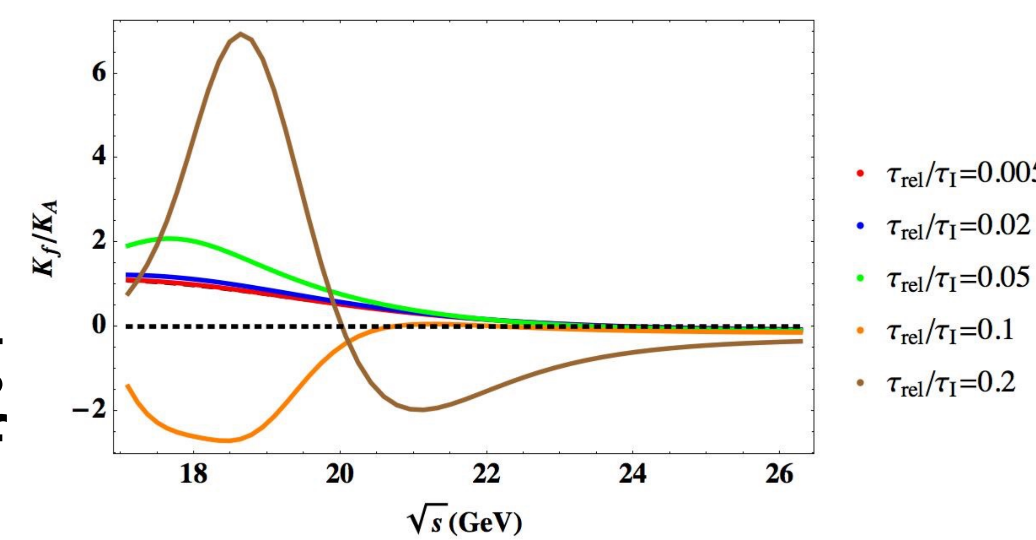




Kurtosis on f. curves for three different positions of f.curves.







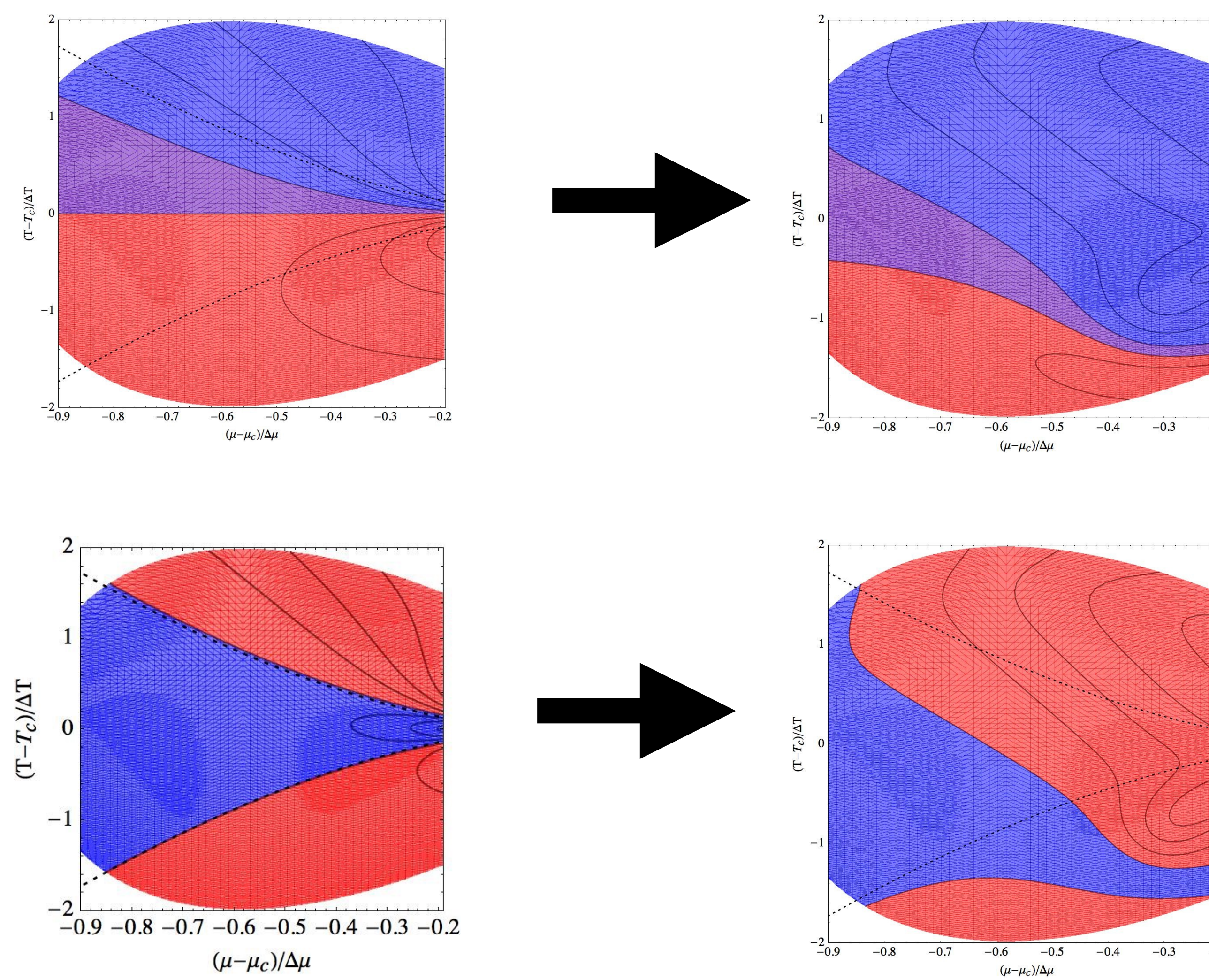
- The flipping of sign of kurtosis is still robust!
- The location that the sign changes depends on non-equilibrium effects.
- The trends in data can be captured by tuning relaxation time and the relative position of freeze-out curve.

Summary I

- We have developed a set of equations to describe the evolution of cumulants in heavy-ion collisions.

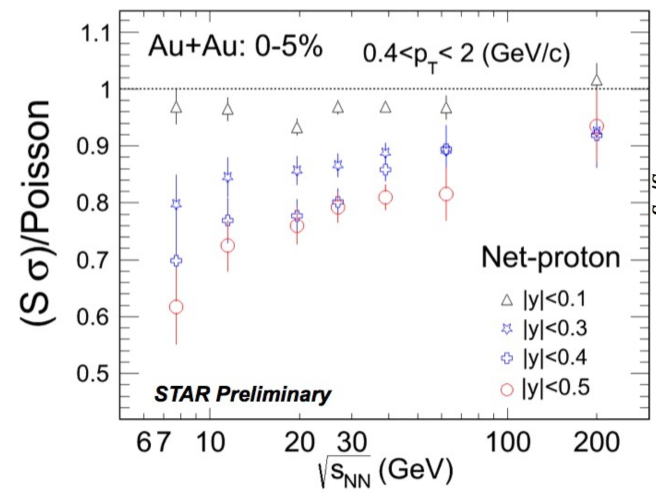
- We illustrate possible complications the would occur in a more comprehensive simulation(mapping between Ising model and QCD, relative position of freeze-out curve, relaxation time etc)

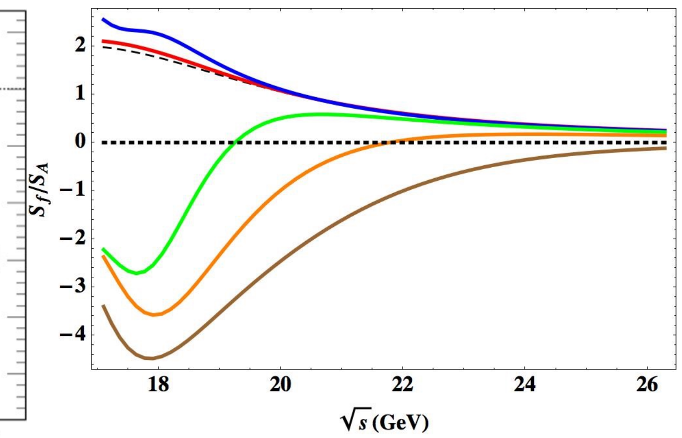
- Regarding the data: keeping non-equilibrium effects in mind are important(such as deformation of the boundary that sign of higher cumulants will change).

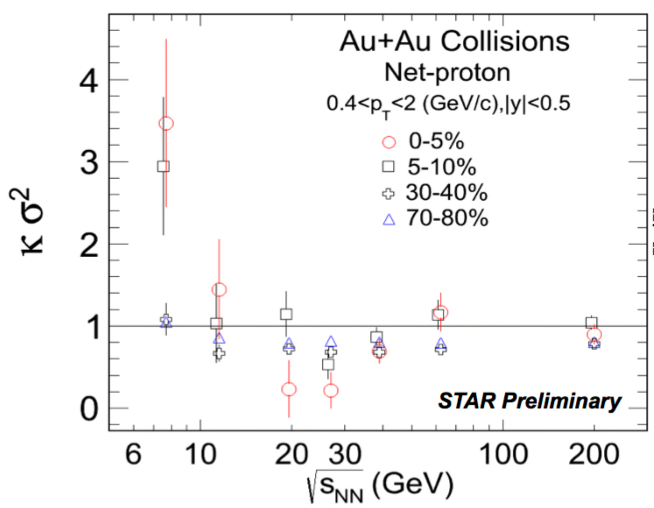


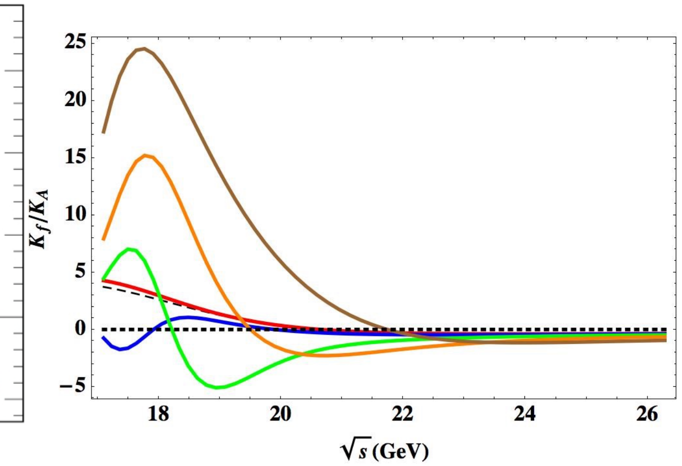
Summary II

- Even in this simple model, results are sensitive to the choice of parameters(relaxation time, relative position of freeze-out curves).
- The parameter space might be constrained by considering correlations among cumulants, finite time scaling among different centrality bins.









- Possibility to reveal dynamical critical properties of QCD in critical regime(similar story at RHIC top energy, not just thermodynamic, also hydrodynamics.)

Back-up Slides